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ON $G(N, p, q)$ SUMMABILITY OF FOURIER SERIES

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NOTATIONS AND DEFINITIONS:

[1.1] The $G(N, p, q)$ transform of $s_n = \sum_{v=0}^{n} a_v$ is defined by

$$t_n = \frac{\sum_{v=0}^{n} p_{n-v} q_v s_v}{r_n}$$

where

$$\gamma_n = \sum_{v=0}^{n} p_v q_{n-v} \; (p - 1 = q - 1 = v - 1 = 0)$$

The series $\sum_{n=0}^{\infty} a_v$ or the sequence $\{s_n\}$ is said to be summable (N, p, q) to s, if $t_n \to \infty$ as $n \to \infty$ and is said to absolutely summable $|N, p, q|$, if tn $\in BV$ and when this happens we shall symbolically by $\{S_n\} \in |N, p, q|$.

We write

$$\varepsilon_n = p_n - p_{n-1} = \Delta p_n$$

$$\xi_n = q_n - q_{n-1} = \Delta q_n$$

$$\mu_n = \delta_n^{\alpha} \text{ where } \delta_n = \sum_{v=0}^{n} q_v$$

and δ_n^α is the n^{th} Cesaro means of the sequence $\{q_n\}$ of order α We noted that

$$\gamma_n = \sum_{v=0}^{n} \epsilon_{n-v}\delta_v$$

and

$$\begin{aligned}\sum_{v=0}^{n} p_{n-v}q_v s_v &= \sum_{r=0}^{n} (p_{n-v} - p_{n-v-1}) \sum_{i=0}^{v} q_i s_i \\ &= \sum_{v=0}^{n} \epsilon_{n-v}t_v\delta_v\end{aligned}$$

Here $\{t_v\}$ is the $(\bar{N}, q)$ mean $([2\theta) \cdot p - 57)$ which is equivalent to $(R, \delta_{n-1}, 1)$ mean $[[2\theta], P - 113)$. Rewriting T_n in term of the simplification given above we know that

$$T_n = \frac{\sum_{v=0}^{n} (p_{n-v} - p_{n-v-1})t_v\delta_v}{\sum_{v=0}^{n} (p_{n-v} - p_{n-v-1})\delta_v}$$

and this form suggest that we can Obtain the following extension of the (N, p, q) method. We now write for any $\{\epsilon_n\}$

$$\begin{aligned}\mathrm{T}_n^\alpha &= \frac{\sum_{v=0}^{n} \epsilon_{n-v}t_v^\alpha\delta_v^\alpha}{\sum_{v=0}^{n} \epsilon_{n-v}\delta_i^\alpha} \\ &= \frac{\sum_{v=0}^{n} \epsilon_{n-v}t_v^\alpha\mu_v}{\sum_{v=0}^{n} \epsilon_{n-v}\mu_v}\end{aligned}$$

where

$$t_n^{(\alpha)} = \frac{1}{\delta_n^\alpha}\sum_{v=0}^{n} (\delta_v - \delta_{v-1})a_v$$

we denote this mean by $G(N,p,q)_\alpha$, $\alpha = 1$

$$t_n^{(1)} = (N,p,q)\ \{S_n\}, \text{ the } G(N,p,q)\alpha$$

method reduces to (N,p,q) method.

We shall also have the occasions.

$$T_n^{\alpha-1} = T_n^* = \frac{\sum_{v=0}^{n} \ \epsilon_{n-v} t_v^{\alpha-1} \mu_v}{\sum_{v=0}^{n} \ \Delta\epsilon_{n-v} \mu_v}$$

$$= \frac{1}{B_n} \sum_{v=0}^{n} \ \Delta\epsilon_{n-v} t_v^{\alpha} \mu_v$$

where

$$B_n = \sum_{v=0}^{n} \ \Delta\epsilon_{n-v} \mu_v$$

[1.2] The object of the present paper is to generalize the theorem of S. DUBEY [1] in the following manner.

THEOREM: Let $G(N,p,q)$ be generalized regular Nörlund method. Let $\{\epsilon_n\}$ and $\{\mu_n\}$ be a non-hegative and non -increasing sequence and

$\lambda_{(t)}$ be a positive non-decreasing function such that

$$\phi(t) = \int_0^t |\phi(u)| du = 0\left(\frac{\lambda\left(\frac{1}{t}\right) \cdot t}{(\epsilon^* u)_t}\right) \qquad (1.2.1)$$

then the Fourier series of $f(t)$ at $t = x$ summable $G(N,p,q)$ to $f(x)$ if

$$\lambda_n \log_n = O((\epsilon^* \mu)_n) \tag{1.2.2}$$

and

$$n|\Delta(\epsilon^* \mu)_n| \leqslant K((\epsilon^* \mu)_n).$$

[1.3] We shall require following result for the proof of our theorem.

LEMMA 1: We have

$$m_n(t) = \frac{1}{(\epsilon^* \mu)_n} \sum_{k=0}^{n} B_k \frac{\sin(n-k)\,t}{t} = 0(1) \tag{1.3.1}$$

and

$$N_n(t) = \frac{1}{(\epsilon^* ())_n} \sum_{k=0}^{n} \beta_k \frac{\sin\left(\frac{n-k+1}{2}\right)}{\sin\frac{1}{\alpha}t} = 0(1) \tag{1.3.2}$$

for $0 \leqslant t \leqslant \frac{1}{n}$.

The lemma follows by the fact that $|\sin t| \leqslant t$.

PROOF OF THE THEOREM:

[1.4] Let $\{s_n\}$ denote the nth partial sum of the Fourier series. We have

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+1/2)t}{2\sin\frac{1}{2}t} dt$$

The generalized Nörlund transform $G(N, p, q)$ of $S_n(x) - f(x)$

$$= \int_0^{\pi} \phi(t)\mathrm{Nn}(t)$$

where

$$N_n(t) = \frac{1}{\pi(\epsilon^*)_n} \sum_{k=0}^{n} B_k \frac{\sin\left(\frac{n-k+1}{2}\right)}{2\sin\frac{1}{2}t}$$

In Order to establish the result. It is sufficient to show that

$$\int_0^{\pi} \phi(t) N_n(t) dt = 0(1) \text{as } n \to \infty \qquad (1.4.1)$$

For $0 < \delta \leqslant 1$, we have

$$\int_0^{\pi} \phi(t)\, \mathrm{N_n}(t)\, dt = \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right) \phi(t) N_n(t) dt$$

$$= A + B + Y \ \text{(Say)}$$

In view of Lemma 1, we get

$$|A| = o\left[n \int_0^{\frac{1}{n}} |\phi(t)| dt\right]$$

$$= o\left[n\phi\left(\frac{1}{n}\right)\right]$$

$$= O\left(\frac{\lambda_n}{(\epsilon^*\mu)_n}\right)$$

$$= O(1) \qquad (1.4.2)$$

by virtue of the hypothesis (1.2.1) and (1.2.2), Now we proceed to estimate B.

We have

$$\int \sum_{k=1}^{n} B_k \sin\left(n-k+\frac{1}{2}\right)t \left| < \sum_{k=1}^{t-1} B_k \sin\left(n-k+\frac{1}{2}\right)t \right| +$$

$$+\sum_{k=n-t+1}^{n} B_k \leqslant \sum_{k=1}^{T} B_k + B_t \max_{T\leqslant\rho\leqslant n-T}\left|\sum_{k=T} \sin\left(n-k+\frac{1}{2}\right)t\right| + \; + \sum_{k=n-t+1}^{n} B_k$$

Since $\{\epsilon_k\}$ and $\{\mu_k\}$ are non-negative and non-increasing. Using the above inequality with the fact that

$$t \leqslant \rho < n-t \sum_{k=t} \sin\left(n-k+\frac{1}{\alpha}\right)t \leqslant t^{-1}$$

we see that

$$|B| \leqslant \frac{k}{(\epsilon^*\mu)_n}\int_{\frac{1}{n}}^{\rho} \frac{|\phi(t)|}{t}\left(\sum_{k=1}^{T-1} B_k\right)dt + \frac{k}{(\epsilon^*\mu)_n}\int_{\frac{1}{n}}^{\rho} \frac{|\phi(t)|}{t^2}(B_t)dt +$$

$$+\frac{k}{(\epsilon^*\mu)_n}\int_{\frac{1}{n}}^{\rho} \frac{|\phi(t)|}{t}\left(\sum_{k=n-t+1}^{n} B_k\right)dt$$

$$= B_1 + B_2 + B_3.$$

We have

$$B_1 \leqslant \frac{k}{(\epsilon^*\mu)_n}\sum_{r=1}^{n-1} \frac{|\phi(t)|}{t}\left(\sum_{k=1}^{T-1} B_k\right)dt$$

$$\leqslant \frac{k}{(\epsilon^{*}\mu)_{n}} \sum_{k=1}^{n-1} \sum_{k=1}^{r} B_{k}(r+1)\left\{\phi\left(\frac{1}{r}\right)-\phi\left(\frac{1}{r+1}\right)\right\}$$

$$\leqslant \frac{k}{(\epsilon^{*}\mu)_{n}} \sum_{k=1}^{n-1} B_{k} \sum_{r=k}^{n} (r+1)\left\{\phi\left(\frac{1}{r}\right)-\phi\left(\frac{1}{r+1}\right)\right\}$$

$$\leqslant \frac{k}{(\epsilon^{*}\mu)_{n}} \sum_{k=1}^{n} B_{k} \left\{\sum_{r=k}^{n} \phi\left(\frac{1}{r}\right)-k\phi\left(\frac{1}{k}\right)\right.$$

$$\left.-(n+1)\,\phi\left(\frac{1}{n+1}\right)\right\}.$$

By changing the order of summation, we get

$$B_{1} \leqslant \frac{k}{(\epsilon^{*}\mu)_{n}} \sum_{r=1}^{n} \phi\left(\frac{1}{r}\right) \sum_{k=1}^{r} B_{k} + \frac{k}{(\epsilon^{*}\mu)_{n}} \sum_{k=1}^{n} B_{k}\left\{k\phi\left(\frac{1}{k}\right)\right\} +$$

$$\left. + (n+1)\,\phi\left(\frac{1}{n+1}\right)\right\}$$

$$= O\left(\frac{(n+1)\phi\left(\frac{1}{n+1}\right)\}}{(\epsilon^{*}\mu)_{n}}\right) \sum_{r=1}^{n} \frac{\lambda(r)}{r(\epsilon^{*}\mu)_{k}} \sum B_{k} +$$

$$+ O\left(\frac{1}{(\epsilon^{*}\mu)_{n}} \sum_{k=1}^{n} B_{k} \frac{\lambda_{(k)}}{(\epsilon^{*}\mu)_{k}} + \frac{\lambda(n+1)}{(\epsilon^{*}\mu)_{n+1}}\right).$$

By virtue of the hypothesis (1.2.2). Since $\{\epsilon_{n}\}$ is non-increasing

$$\epsilon_{n-k} \leqslant \epsilon_{r-k}$$

where $1 \leqslant k \leqslant r \leqslant n$. Hence

$$\mathrm{B}_1 = O\left(\frac{\lambda(n)}{(\epsilon^*\mu)_n}\sum_{r=1}^{n}\frac{1}{r}\right) + o(1)$$

$$= O\left(\frac{\lambda(n)\log n}{(\epsilon^*\mu)_k}\right) + o(1)$$

$$= O(1) \tag{1.4.3}$$

by virtue of (1.2.2)

Next since $\{\epsilon_k\}$ and $\{\mu_k\}$ are non -increasing

$$\mathrm{B}_2 \leqslant \frac{k}{(\epsilon^*\mu_n}\sum_{r=1}^{n}\int_{\frac{1}{r+1}}^{\frac{1}{r}}\frac{|\phi(t)|}{t^2}B_t dt$$

$$\leqslant \frac{k}{(\epsilon^*\mu_n}\sum_{r=1}^{n} r^2 B_r\left\{\phi\left(\frac{1}{r}\right) - \phi\left(\frac{1}{r+1}\right)\right\}.$$

Since r $\epsilon_r\mu_r \leqslant (\epsilon^*\mu_r)_r$ since by the hypo thesis

$$n\Delta(\epsilon^*\mu)_n \leqslant K(\epsilon^*\mu)_n$$

we get

$$\mathrm{B}_2 \leqslant \frac{K}{(\epsilon^*\mu)_n}\sum_{r_{1=2}}^{n}\left((\epsilon^*\mu)_{\theta_2} + (r+1)\Delta(\epsilon^*\mu)_r\right)\phi\left(\frac{1}{r}\right) + k_n\phi\left(\frac{1}{n+1}\right)$$

$$= O\left\{\frac{1}{(\epsilon^*\mu)_n}\sum_{r=2}^{n}\frac{\lambda(r)}{r} + \frac{(r+1)\lambda_{(r)}\Delta(\epsilon^*r)_r}{r(\epsilon^*r)_r}\right\} O\left(\frac{\lambda(n+1)}{(\epsilon^*\mu)_{n+1}}\right)$$

$$= 0\left(\frac{\lambda(n)}{(\epsilon^*\mu)_n}\sum_{r=2}^{n}\frac{1}{r}\right) + O(1).$$

by virtue of (1.2.1) and applying (1.2.2), we see that

$$B_2 = 0(1) \tag{1.4.4}$$

Lastly

$$B_3 \leqslant \frac{k}{(\epsilon^*\mu)_n}\sum_{r=1}^{n-1}\int_{\frac{1}{r+1}}^{\frac{1}{r}}\frac{d(t)1}{t}\left(\sum_{k=n-t+1}^{n} B_k\right)dt$$

$$\leqslant \frac{k}{(\epsilon^*\mu)_n}\sum_{r=1}^{n-1}\sum_{k=n-r}^{n} B_k(r+1)\left\{\phi\left(\frac{1}{r}\right)-\phi\left(\frac{1}{r+1}\right)\right\}$$

$$\leqslant \frac{k}{(\epsilon^*\mu)_n}\sum_{r=1}^{n-1}(\Omega+1)\left\{\phi\left(\frac{1}{r}+\phi\left(\frac{1}{r+1}\right)\right\}\sum_{k=0}^{r} B_k$$

On changing the order of summation we obtain

$$B_3 \leqslant \frac{k}{(\epsilon^*\mu)_n}\sum_{k=1}^{n-1} B_k\sum_{r=k}^{n-1}(r+1)\left\{\phi\left(\frac{1}{r}\right)-\phi\left(\frac{1}{r+1}\right)\right\}+$$

$$+\frac{k}{(e^*\mu)_n}\sum_{r=1}^{n} B_n(r+1)\left\{\phi\left(\frac{1}{r}\right)-\phi\left(\frac{1}{r+1}\right)\right\}$$

It can be seen from the Proof of B_1 that the first term of the above expression is of $O(1)$.

The second term does not exceed

$$\frac{kB_n}{(\in^* \mu)_n}\left(\sum_{k=1}^{n} \phi\left(\frac{1}{r}\right) + 0(1) + (n+1)\phi\left(\frac{1}{n+1}\right)\right) = 0(1)$$

by virtue of the hypothesis (1.2.1) and (1.2.2). Thus,

$$B_3 = O(1) \tag{1.4.5}$$

Lastly by virtue of Riemann Lebesgue

theorem and regularity of the method of summation. We have

$$(y) = \int_{\delta}^{\pi} \phi(t)\big(\epsilon^*()_n t \cdot dt = O(1) \tag{1.4.6}$$

On collecting (1.4.2), (1.4.3), (1.4.4), (1.4.5) and (1.4.6) we obtain (1.4.1). This completes the proof of the theorem.

References

[1] DUBEY, S. : Studies in approximation of function by Fourier series Ph.D. Thesis R.D. Vishwavidyalaya, Jabalpur (1986).